General Equilibrium Notation, Pareto Efficiency

Econ 3030

Fall 2025

Lecture 16

Outline

1. General Equilibrium notation
2. Pareto Optimality
3. Individual Rationality

General Equilibrium Theory

A theory of interactions between all agents in an economy that considers preferences, technology, and resources as the only exogenous items.

1. How would a social planner allocate resources among consumers and firms?
    1. The main concept is Pareto efficiency: outcomes cannot be redistributed to improve welfare.
2. How do competitive markets allocate resources?
    1. The main concept is competitive equilibrium: all agents make *optimal choices taking prices as given*, and prices are such that demand and supply are "equal" in all markets.
3. Important questions:
    1. when is a competitive equilibirum Pareto efficient? *First Welfare Theorem*;
    2. when is a Pareto efficient outcome an equilibirum? *Second Welfare Theorem*;
    3. when does a competitive equilibirum exist? *Arrow-Debreu-McKenzie Theorem*;
    4. when is a competitive equilibrium unique?
    5. is a competitive equilibrium robust to small perturbations of the parameters?
    6. how do we get to a competitive equilibrium, and what happens away from it?
    7. how do we model time and/or uncertainty?

The Economy

- An economy consists of I individuals and J firms who consume and produce L perfectly divisible commodities.
- Each firm $j = 1, ..., J$ is described by a production set $Y_j \subset \mathbb{R}^L$.
- Each consumer $i = 1, ..., I$ is described by four objects:
  - a consumption set $X_i \subset \mathbb{R}^L$;
  - preferences $\succsim_i$ over X_i;
  - an individual endowment, given by a vector of commodities $\boldsymbol{\omega}_i \in X_i$; and
  - a non-negative share $\theta_i \in [0, 1]^J$ of the profits of each firm j.

Definition

An economy with private ownership is a tuple

$$\left\{ \{X_i, \succsim_i, \boldsymbol{\omega}_i, \theta_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$$

- If ownership does not matter, an economy is $\left\{ \{X_i, \succsim_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \boldsymbol{\omega} \right\}$, where $\boldsymbol{\omega} = \sum_{i=1}^{I} \boldsymbol{\omega}_i \in \mathbb{R}^L$ is the aggregate endowment.
  - The aggregate endowment describes the resources available to the economy.

Allocations

- What is an outcome for the economy?

Definition

An allocation specifies a consumption vector $\mathbf{x}_i \in X_i$ for each consumer $i = 1, ..., I$, and a production vector $\mathbf{y}_j \in Y_j$ for each firm $j = 1, ..., J$; an allocation is

$$(\mathbf{x}, \mathbf{y}) \in \prod_{i=1}^{I} X_i \times \prod_{j=1}^{J} Y_j \subset \mathbb{R}^{L(I+J)}$$

- This is a long lists of what everyone consumes and what every firm produces:

$$(\mathbf{x}_1, ..., \mathbf{x}_I, \mathbf{y}_1, ..., \mathbf{y}_J) \in X_1 \times ... \times X_I \times Y_1 \times ... \times Y_J$$

  - consumption must be in each consumer's preference domain ($\mathbf{x}_i \in X_i$), and
  - production must be technologically feasible for each firm ($\mathbf{y}_j \in Y_j$).
- Are all allocations achieveable? No
  - The definition of allocation does not reflect available resources.
- The initial endowments place constraints on what can be achieved.

Feasibility

Definition

An allocation $(\mathbf{x}, \mathbf{y})$ is feasible if

$$\sum_{i=1}^{I} \mathbf{x}_i \leq \sum_{i=1}^{I} \boldsymbol{\omega}_i + \sum_{j=1}^{J} \mathbf{y}_j$$

- A feasible allocation can be consumed and produced given the economy's resources: consumption and production are compatible with the aggregate endowment.
  - Feasibility has nothing to do with choices by consumers or firms.
  - Feasibility has nothing to do with ownership.
- Why the inequality instead of an equality?

Definition

The set of feasible allocations is denoted by

$$\{(\mathbf{x}, \mathbf{y}) \in \prod_{i=1}^{I} X_i \times \prod_{j=1}^{J} Y_j : \sum_{i=1}^{I} \mathbf{x}_i \leq \boldsymbol{\omega} + \sum_{j=1}^{J} \mathbf{y}_j\} \subset \mathbb{R}^{L(I+J)}$$

Conditions for Existence of Feasible Allocations

Proposition

If the following conditions are satisfied, the set of feasible allocations is closed and bounded, and non-empty.

1. X_i *is closed and bounded below for each* $i = 1, ..., I$.
2. Y_j *is closed for each* $j = 1, ..., J$.
3. $Y = \sum_j Y_j$ *is convex and satisfies*
    1. $\mathbf{0}_L \in Y$ *(inaction),*
    2. $Y \cap \mathbf{R}_+^L \subseteq \{\mathbf{0}_L\}$ *(no free-lunch), and*
    3. *if* $\mathbf{y} \in Y$ *and* $\mathbf{y} \neq \mathbf{0}_L$ *then* $-\mathbf{y} \notin Y$ *(irreversibility).*
4. $-\mathbb{R}_+^L \subset Y_j$ *(free-disposal), and there are* $\mathbf{x}_i \in X_i$ *for each* i *such that* $\sum_i \mathbf{x}_i \leq \boldsymbol{\omega}$

- We will take the above properties of the set of feasible allocations for granted unless stated otherwise.

Edgeworth Box Economy

- No production. Two goods, denoted 1 and 2, and two consumers, denoted A and B.
- An allocation is given by four numbers

$$\mathbf{x}_A = (x_{1\,A}, x_{2\,A}) \text{ and } \mathbf{x}_B = (x_{1\,B}, x_{2\,B})$$

- The set of feasible allocations is described as

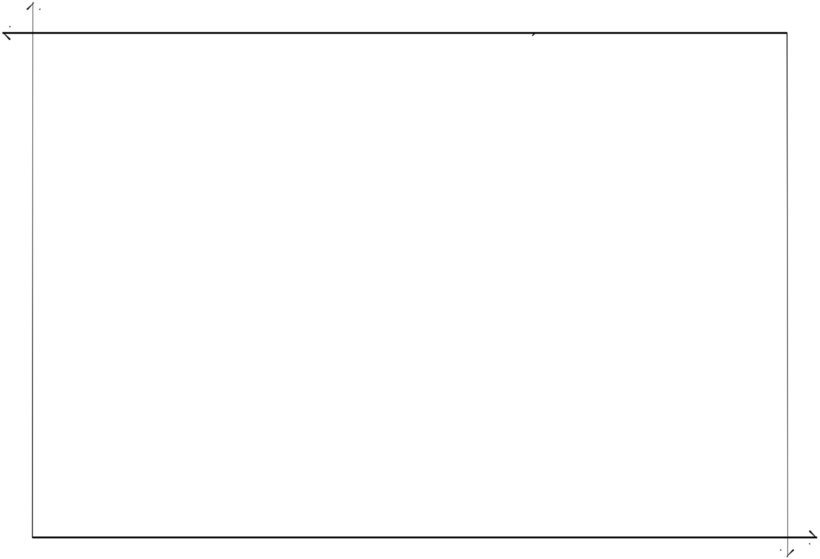
$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}^4 : \mathbf{x}_A \in \mathbb{R}^2_+, \mathbf{x}_B \in \mathbb{R}^2_+, \text{ and } \mathbf{x}_A + \mathbf{x}_B \leq \boldsymbol{\omega}_A + \boldsymbol{\omega}_B\right\}$$

  so a feasible allocation must satisfy

$$x_{1\,A} + x_{1\,B} \leq \omega_{1\,A} + \omega_{1\,B} \qquad \text{and} \qquad x_{2\,A} + x_{2\,B} \leq \omega_{2\,A} + \omega_{2\,B}$$

- Since the inequality is important, we can imagine the economy has one firm with the following technology: $Y_1 = -\mathbb{R}^2_+$. This firm can destroy any amount of the goods.
- We can represent all feasible allocation by means of an Edgeworth Box.

Edgeworth Box



Exchange Economy

- In an exchange economy there is no production.
  - This is like an Edgeworth Box economy, but there are I consumers and L goods.
  - There is a single firm that can dispose of amounts of the goods: formally, $J = 1$, and $Y_j = -\mathbb{R}^L_+$.

In an exchange economy the set of feasible allocations is

$$\{\mathbf{x} \in \prod_{i=1}^{I} X_i : \sum_{i=1}^{I} \mathbf{x}_i \leq \boldsymbol{\omega}\} \subset \mathbb{R}^{LI}$$

- This is a simple environment in which many results are easier to prove, and that most of the time has the same intuition as an economy with production.

Edgeworth Box

- An Edgeworth Box is an exchange economy with two consumers and two goods.
- Formally $L = 2$, $I = 2$, $X_1 = X_2 = \mathbb{R}^2_+$, $J = 1$, and $Y_j = -\mathbb{R}^2_+$.

Representative Agent Economy

- There are two goods, food F and labor L, one firm, and one consumer (both called *Agent*).
- The *Agent*'s endowment is $\boldsymbol{\omega} = (1, 0)$ (one unit of labor and no food).
- The *Agent*'s technology transforms labor into food:

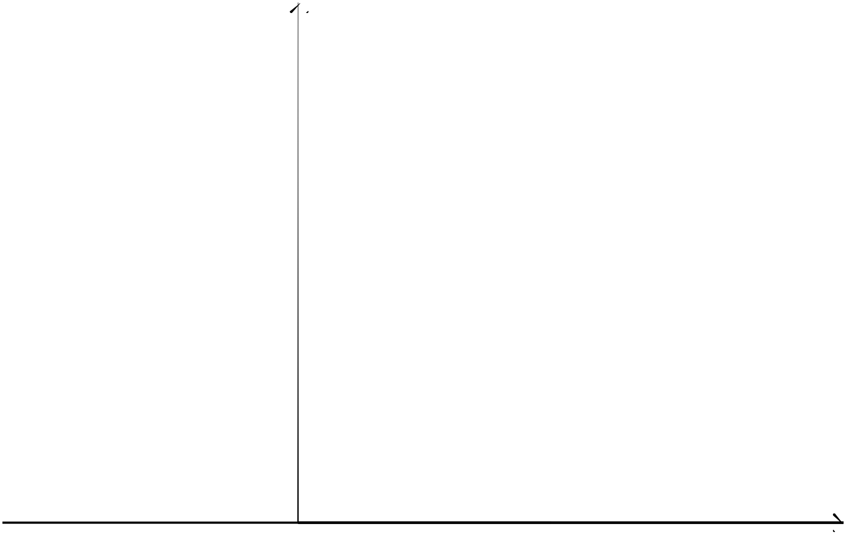
$$Y = \left\{ (y_L, y_F) \in \mathbb{R}^2 : y_L \leq 0, \text{ and } y_F \leq f(-y_L) \right\}$$

  - The function $f(\cdot)$ is a standard production function.
    - The technology has the free disposal property: if $\mathbf{y} \in Y$ and $\mathbf{y}' \leq \mathbf{y}$ then $\mathbf{y}'$ is also an element of Y.
- A feasible allocation is described as

$$\left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 : \mathbf{x} \in \mathbb{R}^2_+, \ \mathbf{y} \in Y, \text{ and } \mathbf{x} \leq \boldsymbol{\omega} + \mathbf{y} \right\}$$

  - Because of free disposal, one can think of the inequality as an equality without loss.
- Draw the production possibility set and the set of feasible consumption bundles.

Representative Agent



Efficiency

- What does it mean for an outcome to achieve a "minimal" notion of efficiency?
- First, an efficient outcome should be feasible.
  - Outcomes that are not feasible may be great, but are not interesting because they cannot be achieved.
- Second, an outcome is definitely not efficient if there exist another outcome that nobody would object switching to, and that someone would like better.
  - If such a different outcome exists the current outcome is not using the economy's resources in the best possible way.
- Efficiency is not about anyone's choices or decisions, but just about conditions under which one can say that an outcome is acceptable in a specific way.

Pareto Optimality (a.k.a. Pareto Efficiency)

Definition

An allocation $(\mathbf{x}', \mathbf{y}')$ Pareto dominates the allocation $(\mathbf{x}, \mathbf{y})$ if

$$\mathbf{x}_i' \succsim_i \mathbf{x}_i \text{ for all } i = 1, \ldots, I \quad \text{and} \quad \mathbf{x}_i' \succ_i \mathbf{x}_i \text{ for some } i$$

Definition

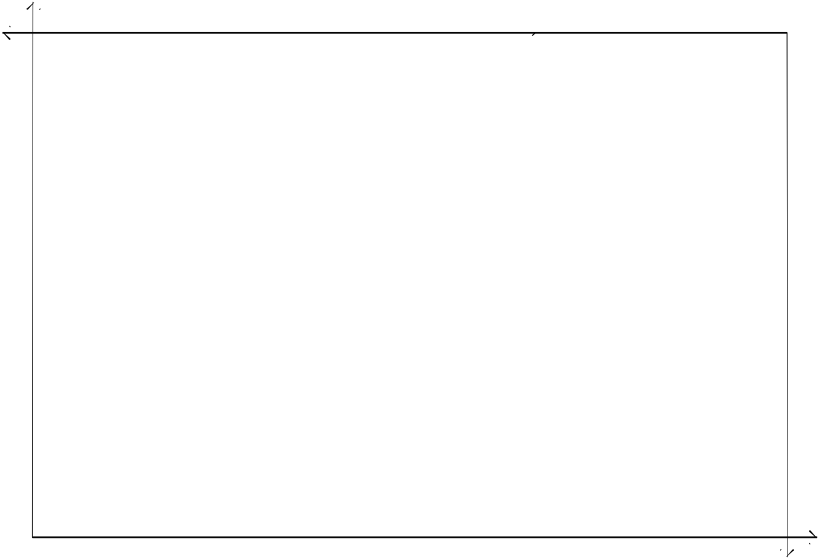
A feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto optimal if there is no other *feasible* allocation that Pareto dominates it; that is, there exists no feasible $(\mathbf{x}', \mathbf{y}')$ such that

$$\mathbf{x}_i' \succsim_i \mathbf{x}_i \text{ for all } i = 1, \ldots, I \quad \text{and} \quad \mathbf{x}_i' \succ_i \mathbf{x}_i \text{ for some } i$$

- Efficiency means there is no way to benefit someone without harming anyone.
- Feasibility is necessary: a feasible allocation is Pareto optimal if no feasible allocation Pareto dominates it.
- We only consider consumers, but *firms matter because of what they produce*.

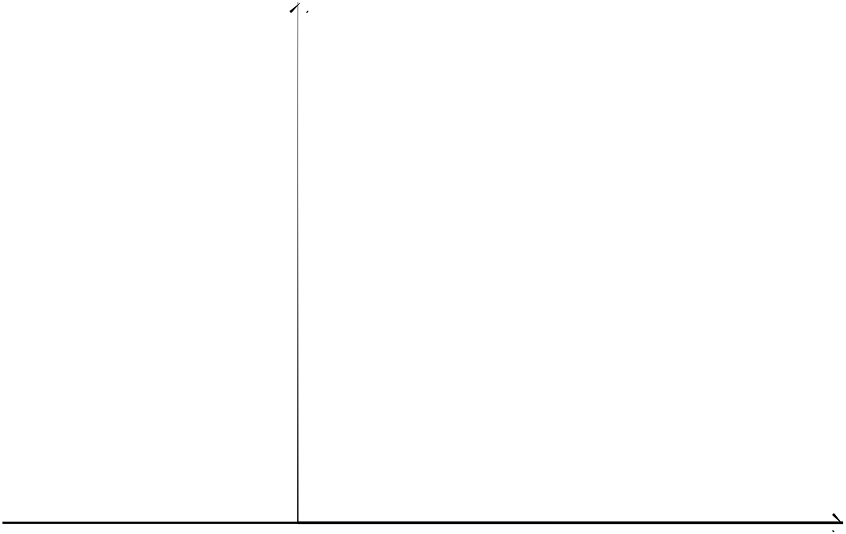
Pareto Optimality: Examples

- Draw a Pareto optimal allocation for an Edgeworth box economy.



Pareto Optimality: Representative Agent

- Draw a Pareto optimal allocation for a Representative Agent economy.



Pareto Optimality Is Not Fairness

- Pareto optimal allocations are not necessarily fair:
  - when preferences are monotone, the allocations that give the aggregate endowment to a single individual are Pareto efficient.
- Fairness is hard to tackle and we will mostly ignore it.

- Pareto optimal allocations also disregard what each individual owns to begin with.
- The constraint that individuals should not lose relative to where they start from is easier to tackle: make sure nobody is better-off alone.

Individual Rationality

- Suppose each individual has veto power over allocations.

Observation

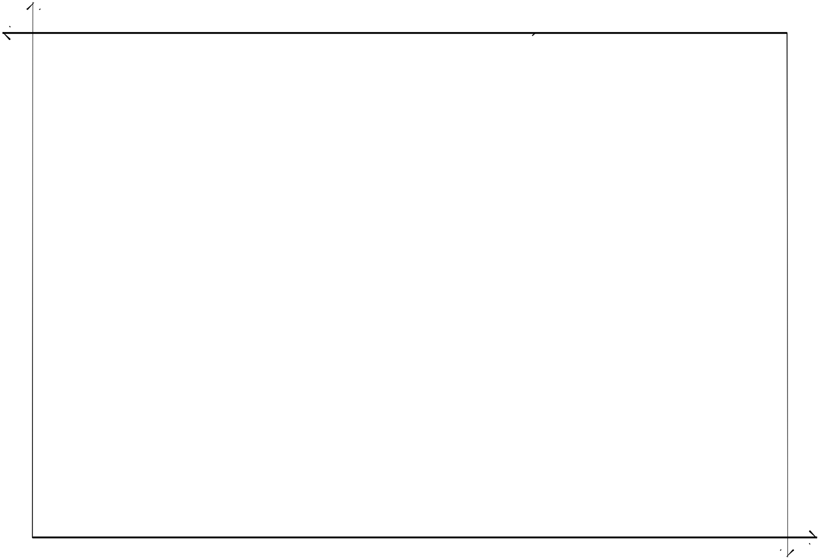
- Any consumer could veto allocations that do not improve her initial conditions.
- She rejects allocations that are worse than her initial endowment.
  - She can, for example, refuse to participate in the economy in those cases.

Definition

A feasible allocation $(\mathbf{x}, \mathbf{y})$ is individually rational if $\mathbf{x}_i \succsim_i \boldsymbol{\omega}_i \quad$ for all i.

- An individually rational allocation represents a trade that does not make anyone worse-off relative to their initial endowment.
- Pareto optimal allocations are not necessarily individually rational.
- Draw a picture of Pareto optimal allocations that are not individually rational.

Pareto Optimality and Individual Rationality



Next Class

- Social Welfare Functions
- Utility Possibility Set
- Planner's Problem and Pareto Optimality